Homework 6 Solutions

Math 131B-1

- (1.27) a) $-2+2i$ b) $-\frac{6}{25}+\frac{17}{25}i$ c) $1+i$ d) $1+i$

- (1.30) a) The unit circle. b) An open disk of radius 1 about the origin. c) A closed disk of radius 1 about the origin. d) The vertical line $\{\frac{1}{2}+bi : b \in \mathbb{R}\}$. e) The horizontal line $\{a+\frac{1}{2}i : a \in \mathbb{R}\}$. f) A circle of radius 1 centered at $1+0i$. (The equation for f reduces to $2a = a^2+b^2$, or $1 = (a-1)^2+b^2$.)

- (1.36) Axiom 6 holds: If two complex numbers z, w have the same norm and argument, then $z = w$; otherwise it is possible to decide that $z > w$ or $w > z$.

  Axiom 7 does not hold: In this relation $1+i < -1+i$, but adding $-2i$ to both sides does not preserve the relation, since $-1-i < 1-i$.

  Axiom 8 holds: Everything which is not 0 is greater that 0, so if $z, w > 0$, since $zw \neq 0$, we have $zw > 0$.

  Axiom 9 holds: Given $z > w$ and $w > u$, either they all have the same norm, in which case $arg(z) > arg(w) > arg(u)$, or they do not, in which case $|z| > |u|$. In either case $z > u$.

- Let $z = a+bi$. Then $|R(z)| = |a| = \sqrt{a^2} \leq \sqrt{a^2+b^2} = |z|$. Similarly $|I(z)| = |b| \leq |z|$. However, notice that $|z|^2 = a^2+b^2 \leq a^2+2|a||b|+b^2 = (|R(z)|+|I(z)|)^2$, so $|z| \leq |R(z)|+|I(z)|$. Finally, if $z = a+bi$, $w = c+di$, we have

$$\begin{aligned}
|z+w|^2 = (a+c)^2+(b+d)^2 &= a^2+2ac+c^2+b^2+2bd+d^2 \\
&\leq a^2+2|a||c|+c^2+b^2+2|b||d|+d^2 \\
&= (a^2+b^2)+2\sqrt{a^2c^2+b^2d^2}+(c^2+d^2) \\
&\leq (a^2+b^2)+2\sqrt{(a^2+b^2)(c^2+d^2)}+(b^2+d^2) \\
&= (\sqrt{a^2+b^2}+\sqrt{c^2+d^2})^2 \\
&= (|z|+|w|)^2
\end{aligned}$$

  Therefore $|z+w| \leq |z|+|w|$.

- We use what we know about the convergence of the Cauchy product.

$$\begin{aligned}
\cos(x+y) &= \sum_{n=0}^{\infty} \frac{(-1)^n (x+y)^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{2n} \frac{(2n)!}{k!(2n-k)!} \frac{(-1)^n x^k y^{2n-k}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{2n} \frac{x^k}{k!} \frac{y^{2n-k}}{(2n-k)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \left[\sum_{m=0}^{n} \frac{x^{2m}}{(2m)!} \frac{y^{2n-2m}}{(2n-2m)!} + \sum_{m=0}^{n-1} \frac{x^{2m+1}}{(2m+1)!} \frac{y^{2n-2m-1}}{(2n-2m-1)!} \right] \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{x^{2m}}{(2m)!} \frac{y^{2n-2m}}{(2n-2m)!} + (-1)^n \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} \frac{x^{2m+1}}{(2m+1)!} \frac{y^{2n-2m-1}}{(2n-2m-1)!} \\
&= \cos(x)\cos(y) - \sin(x)\sin(y)
\end{aligned}$$

The only thing to check is that the sign on the Cauchy product of the two cosines is $(-1)^m(-1)^{n-m} = (-1)^n$ and the sign on the Cauchy product of the two sines is $(-1)^m(-1)^{n-m-1} = (-1)^{n-1} = -(-1)^n$, accounting for the negative sign in the final expression. The other equality is similar.

- We take the Cauchy product of the power series expansions of the functions in question.
- Suppose not, then there exist x arbitrarily close to x_0 such that $f(x_0) = 0$. In particular, there exists x_n such that $0 < |x_n - x_0| < \frac{1}{n}$ but $f(x_n) = 0$. Then since $x_n \to x_0$, we have $\frac{f(x_n)-f(x_0)}{x_n - x_0} \to f'(x_0)$. But $f(x_n) - f(x_0) = 0$, so this implies $f'(x_0) = 0$, a contradiction. Ergo there is some c such that $0 < |x - x_0| < c$ implies $f(x) \neq 0$. In particular, there is some c such that $\sin x$ is nonzero on $(0, c)$.
- Since we found the derivatives of sine and cosine in class, by the quotient rule $\frac{d}{dx}(\tan x) = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos^2 x - (-\sin^2 x)}{\cos^2 x} = 1 + \tan^2 x$. In particular, the derivative exists and is strictly positive on $(-\frac{\pi}{2}, \frac{\pi}{2})$ because cosine is nonzero on this range. The derivative of the inverse function $g(x) = \tan^{-1}(x)$ is $g'(x) = \frac{1}{f'(g(x))} = \frac{1}{1+\tan^2(\tan^{-1}(x))} = \frac{1}{1+x^2}$.
- If f is 1-periodic, then $f(\mathbb{R}) = f([0,1])$. Since f is continuous and $[0,1]$ is a closed interval, we see that $f([0,1])$ is bounded, so $f(\mathbb{R})$ is as well. But if we do not assume f is continuous, we could take some example such as $f(0) = 1 = f(1)$ and $f(x) = \frac{1}{x}$ on $(0,1]$ and extend 1-periodically to get an unbounded 1-periodic function. (Even though f is continuous on $(0,1]$.)